Comparative Analysis of Order Reduction Techniques

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Abstract— **The mathematical models used to represent physical processes are generally large and complex. Since simpler models are commonly employed in control, design and analysis, it is required to reduce the order of complex systems. This paper presents a comparative analysis of the performance of four order reduction methods. The computation of the lower order transfer functions is based on balance truncation algorithm, Hankel operator, moment matching and optimization algorithm. To demonstrate the performance of the reduction methods, second and third order transfer functions are computed for a plant model of fractional order. The integer transfer function of the plant model is obtained by applying Oustaloup filter approximation technique. For comparison of the reduction methods with ideal response, the analysis has been done both in frequency-domain and time-domain using MATLAB.**

Keywords- complex systems; order reduction; lower order model

I. INTRODUCTION

Order reduction means the original system is approximated to a comparatively lower order, while still capturing the ideal features of the original system. As is known for simulation and control, the physical processes are modeled mathematically using differential equations. But the demand for high accuracy, leads to large and complex models specifically for interconnected systems and for integer models, of fractional order systems. For this reason, order reduction has been one of the important areas of research. The application areas of order reduction are control systems, integrated circuits, micro electromechanical systems (MEMS), and a few more. The issue of system approximation has two parts: approximation procedure and suitable error calculation system. This requires solving complex system using linear algebra and various numerical techniques. The order reduction methods available in literature are based on different concepts viz. projection based, norm based, s-domain based, matching moments based, aggregation based, error minimization based etc and is used in wide number of applications for reducing the order of large systems [1-9]. The approximation errors in the frequency domain are mostly represented using the matrix norm 2 and infinity.

In this paper, Balanced truncation method, Matched DC gain method, Pade approximation method and sub-optimum method have been used to compute the lower order models. As an example, the s-domain transfer function of a plant having fractional power is approximated to 6^{th} order transfer function using Oustaloup filter approximation technique and then it is reduced to second and third order transfer functions using the four reduction methods. The performance of the lower order transfer functions are compared with the 6^{th} order transfer function and also with the ideal response of the plant. Comparisons are performed by plotting magnitude versus frequency plots as well as magnitude versus time plots of the plant model, its integer transfer function and second & third order transfer functions

The content of the paper in the next sections is organized as: Oustaloup approximation technique and the order reduction methods used in this paper are briefed in Section II & III respectively. The lower order models of a fractional order plant are presented in Section IV and its performance discussed in Section V. Conclusion is given in Section VI.

II. OUSTALOUP APPROXIMATION TECHNIQUE

The steps for the Oustaloup approximation technique are:

- Frequency band $[\omega_1, \omega_2]$ rad/sec is set; ω_1 is the lower frequency and ω_2 is the upper frequency.
- $s^{\pm\alpha}$, the fractional operator is approximated [10].

The fractional operator is

$$F(s) = s^{\pm\alpha}\ ;\ (0 < \alpha < 1) \quad (1)$$

The Oustaloup method based approximation having integer orders is

$$F_O(s) = \left(\frac{\omega_g}{\omega_2}\right)^{\alpha} \prod_{n=-N}^{N} \left(\frac{1+s/z_n}{1+s/p_n}\right) \quad (2)$$

where number of poles $p_n = \omega_1 \left(\frac{\omega_2}{\omega_1}\right)^{\frac{\left(n+N+\frac{1}{2}+\frac{\alpha}{2}\right)}{(2N+1)}}$;

number of zeros $z_n = \omega_1 \left(\frac{\omega_2}{\omega_1}\right)^{\frac{\left(n+N+\frac{1}{2}-\frac{\alpha}{2}\right)}{(2N+1)}}$;

unit gain frequency $\omega_g = (\omega_1\omega_2)^{1/2}$; &

natural number N

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III. ORDER REDUCTION

The s-domain representation of a finite dimensional filter in the transfer function form is

$$F(s)=\frac{Y(s)}{U(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots a_0} \quad (3)$$

and the state space equations in the transfer function form is written as

$$F(s)=\frac{Y(s)}{U(s)}=C\left(sI-A\right)^{-1}B+D \quad (4)$$

The four order reduction methods used in this paper are briefly discussed here. These are:

A. Balanced truncation method

The steps for order reduction using this method are:

- Hankel singular values are determined.
- Grammians for controllability and observability are evaluated.
- Schur balance truncation algorithm [11] is implemented.

B. Matched DC gain mehod

The steps for order reduction using this method are:

- Hankel Singular Value decomposition [12] is performed. For the state space model,

$$\begin{aligned}\dot{x}&=\bar{A}\,\bar{x}+\bar{B}\,u\\ y&=\bar{C}\,\bar{x}+D\,u\end{aligned} \quad (5)$$

where $\dot{x}=\begin{bmatrix}\dot{x}_1\\ x_2\end{bmatrix}$; $A=\begin{bmatrix}A_{11} & A_{21}\\ A_{21} & A_{22}\end{bmatrix}$; $x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$; $B=\begin{bmatrix}B_1\\ B_2\end{bmatrix}$ and $C=\begin{bmatrix}C_1 & C_2\end{bmatrix}$

Substituting $\dot{x}_2=0$, x_1 is obtained as

$$\begin{aligned}\dot{x}_1&=\left[A_{11}-A_{12}A_{22}^{-1}A_{21}\right]x_1+\left[B_1-A_{12}A_{22}^{-1}B_2\right]u\\ y&==\left[C_1-C_2A_{22}^{-1}A_{21}\right]x+\left[D-C_2A_{22}^{-1}B_2\right]u\end{aligned} \quad (6)$$

The limitation of this method is that A_{22} must be invertible.

C. Pade approximation method

The steps involved for order reduction using this method are

- The higher-order transfer function is represented in terms of moments.
- Moment matching is performed corresponding to the desired-order transfer function [13-14].

D. Sub-optimum method

The steps involved for order reduction using this method are

- An error function is defined
- Powell's optimization algorithm [15] is implemented.

IV. PLANT MODEL

In this section, a plant model with fractional powers is considered for simulation purposes. [16]. The four reduction techniques are applied and the reduced integer models of the fractional order plant is obtained and the frequency and step responses are compared.

$$F(s)=\frac{3}{2s^{1.3}+1} \quad (7)$$

Equation (7) is approximated to 6th order integer transfer function $F_{INT}(s)$ using Oustaloup approximation technique with $\omega_1=10^{-3}$ & $\omega_2=10^3$ and N=2.

$$F_{INT}(s)=\frac{3s^5+1217s^4+2.932e4s^3+4.438e4s^2+4221s+23.83}{15.89s^6+2815s^5+2.999e4s^4+2.932e4s^3+1.56e4s^2+1409s+7.943} \quad (8)$$

Since the order of this plant is high, it is reduced using the four reduction techniques to obtain second and third order plant models as follows:

- 2nd order models
  - Balanced truncation method

$$F_{ORDER2\ BT}(s)=\frac{0.8197s+1.64}{s^2+0.9334s+0.5173} \quad (9)$$

  - Matched DC gain method

$$F_{ORDER2\ MDG}(s)=\frac{-0.1692s^2+1.241s+0.8991}{s^2+0.6903s+0.2997} \quad (10)$$

  - Pade approximation method

$$F_{ORDER2\ PADE}(s)=\frac{4.575s+0.0321}{s^2+1.528s+0.0107} \quad (11)$$

  - Sub optimum method

$$F_{ORDER2\ SOP}(s)=\frac{1.184s+0.8862}{s^2+0.6601s+0.2954} \quad (12)$$

- 3rd order models
  - Balanced truncation method

$$F_{ORDER3\ BT}(s)=\frac{0.7513s^2+2.251s+0.5526}{s^3+1.466s^2+0.9055s+0.1817} \quad (13)$$

  - Matched DC gain method

$$F_{ORDER3\ MDG}(s)=\frac{-0.04063s^3+0.9053s^2+1.602s+0.1312}{s^3+s^2+0.5548s+0.04373} \quad (14)$$

  - Pade approximation method

$$F_{ORDER3\ PADE}(s)=\frac{1.982s^2+0.2352s+0.001345}{s^3+0.7064s^2+0.07852s+0.0004485} \quad (15)$$

  - Sub optimum method

$$F_{ORDER3\ SOP}(s)=\frac{0.8662s^2+1.652s+0.1372}{s^3+1.012s^2+0.5723s+0.04572} \quad (16)$$

V. PERFORMANCE AND DISCUSSIONS

The time and frequency domain analysis is performed for the fractional order plant model for both the orders. The step and frequency response of both orders are plotted in Figs. 1,2 and 3,4 respectively. The magnitude and phase error plots are also shown in Figs. 5 & 6. The observations are as follows:

- Time domain response
  - From Fig. 1, for the second order approximations the second order Pade approximated plant model gives best results. The rise time is 1.4287 sec, settling time is 2.4877 sec and peak overshoot is 0.0071
  - From Fig. 2, for the third order approximations the peak overshoot (0.1854) for the third order Pade approximated plant model is least; settling time of third order Balanced truncation approximated plant model is 10.1459 sec, which is best among all four, though its rise time is slightly higher than that Sub optimum approximated plant model which is 2.3395 sec.

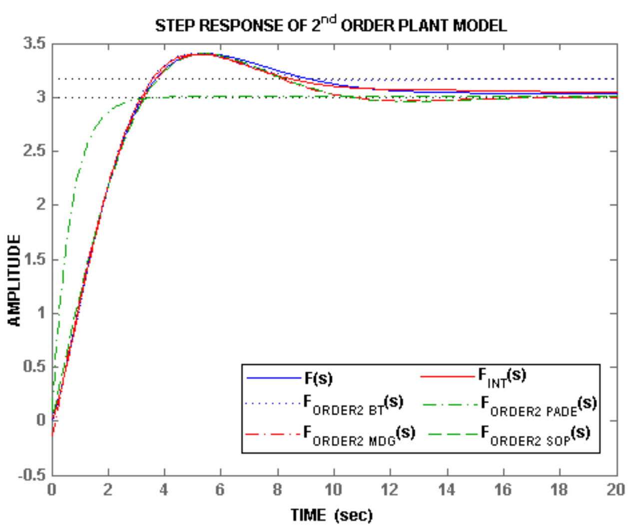

Figure 1. Step responses of second order plant models

- Frequency domain response
  - From Figs. 3 & 5, it is seen that in the entire frequency range $[10^{-2}\ 10^{2}]$ rad/sec, for the second order approximations both magnitude and phase response of Balanced truncation approximated plant model tracks the ideal trace of F(s). The maximum magnitude error is 0.0545 dB at 100 rad/sec frequency and maximum phase error is 25.4714 degrees at 20.61 rad/sec frequency.

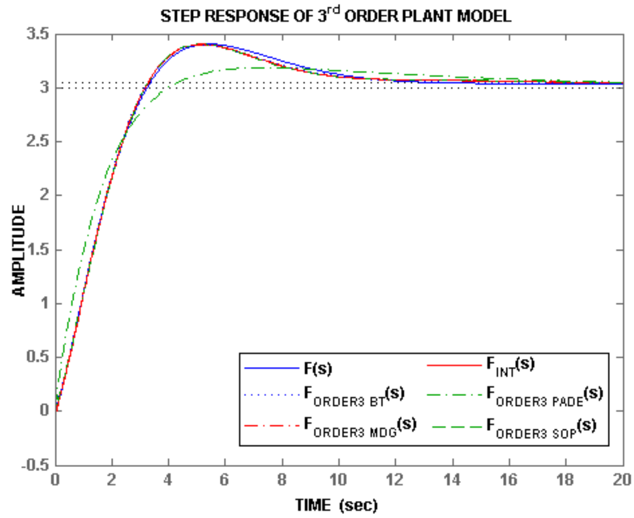

Figure 2. Step responses of third order plant models

- From Figs. 4 & 6, it is observed that in the entire frequency range $[10^{-2}\ 10^{2}]$ rad/sec for the third order approximations the magnitude of Balanced truncation approximated plant model tracks the ideal trace of F(s) and the phase of Matched DC gain approximated plant model matches closely with the ideal phase of F(s) The maximum magnitude error is 0.0057 dB at 100 rad/sec frequency and maximum phase error is 24.2367 degrees at 22.41 rad/sec frequency.

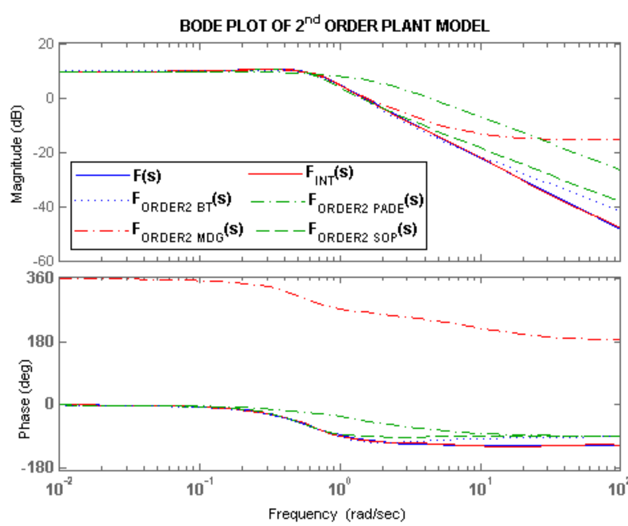

Figure 3. Bode plots of second order plant models

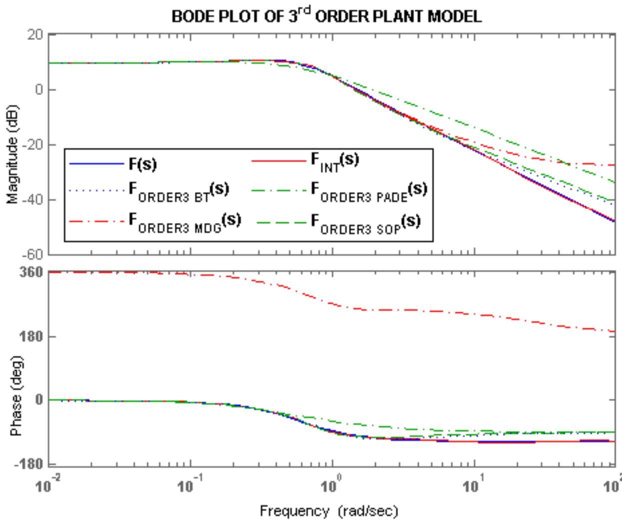

Figure 4. Bode plots of third order plant models

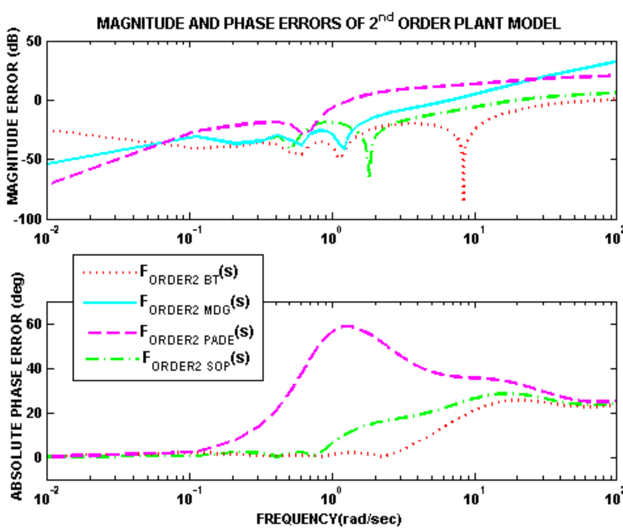

Figure 5. Error plots of second order plant models

VI. Conclusion

In this paper, a plant model with fractional power is transformed to a rational transfer function using Oustaloup filter approximation technique. It is further approximated to second and third order transfer functions using the four order reduction methods. The performance of lower-order transfer functions are compared with each other and also with the higher-order rational transfer function. From the frequency and time domain results of the chosen fractional order plant, it can be inferred that Pade and Balanced truncation approximated lower-order models perform better than the other approximated lower-order models.

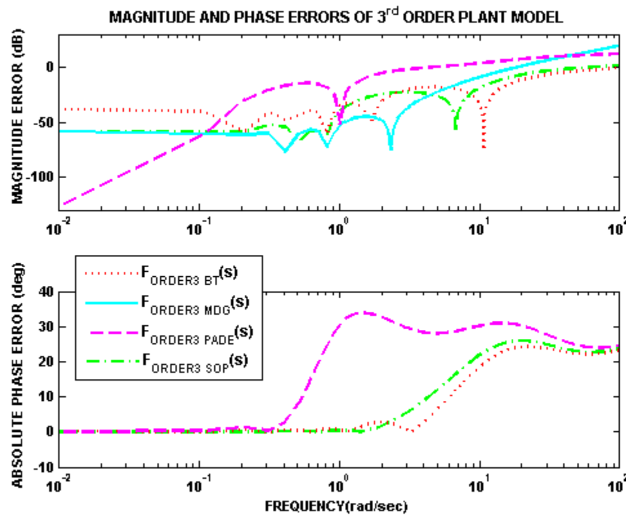

Figure 6. Error plots of third order plant models

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